

# QUANTIZATION OF LIE BIALGEBRAS, PART VI: QUANTIZATION OF GENERALIZED KAC-MOODY ALGEBRAS

Pavel Etingof<sup>1</sup> and David Kazhdan<sup>2</sup>

To Bert Kostant with admiration

## 1. INTRODUCTION.

This paper is a continuation of the series [EK1-5]. We show that the image of a Kac-Moody Lie bialgebra with the standard quasitriangular structure under the quantization functor defined in [EK1,EK2] is isomorphic to the Drinfeld-Jimbo quantization of this Lie bialgebra, with the standard quasitriangular structure. This implies that when the quantization parameter is formal, then the category  $\mathcal{O}$  for the quantized Kac-Moody algebra is equivalent, as a braided tensor category, to the category  $\mathcal{O}$  over the corresponding classical Kac-Moody algebra, with the tensor category structure defined by a Drinfeld associator. This equivalence is a generalization of the functor constructed in [KL].

In particular, we answer positively questions 8.1, 8.2 from [Dr1]: we show that the characters of irreducible highest weight modules for a quantized Kac-Moody algebra  $\mathfrak{g}$  are the same as in the classical case, and that the quantum deformation of the appropriate completion  $\hat{U}(\mathfrak{g})$  of  $U(\mathfrak{g})$  is trivial as a deformation of algebras.

Moreover, our results are valid for the Lie algebra  $\mathfrak{g}(A)$  corresponding to any symmetrizable matrix  $A$  (not necessarily with integer entries). This answers question 8.3 in [Dr1], of Drinfeld and Gelfand (how to define a flat deformation  $U_{\hbar}(\mathfrak{g}(A))$  of the Hopf algebra  $U(\mathfrak{g}(A))$  for any symmetrizable matrix  $A$ ).

We also prove the Drinfeld-Kohno theorem for the algebra  $\mathfrak{g}(A)$  (it was previously proved by Varchenko [V] using integral formulas for solutions of the KZ equations).

**Remark.** One of the important facts used in this paper is that the quantization functors from [EK1,EK2] commute with duals and doubles. However, the original version of this paper, which appeared in 2000, unfortunately did not contain a convincing proof of this statement. Namely, it referred to [EK1,EK2,EK3], where this fact was proved for finite dimensional Lie bialgebras, and claimed that in general the proof was similar. But it turns out that actually, it is not easy to extend the argument of [EK1,EK2,EK3] to the general case, since it uses cyclic expressions. A general proof of this result was obtained by Enriquez and Geer in 2007, see [EG]. This revised version of our paper takes this fact into account, and replaces insufficient references to [EK1,EK2,EK3] by the reference to [EG].

<sup>1</sup>Department of Mathematics, 2-165, MIT, Cambridge, MA 02139, USA, etingof@math.mit.edu

<sup>2</sup>Einstein Institute of Mathematics Edmond J. Safra Campus, Givat Ram The Hebrew University of Jerusalem Jerusalem, 91904, Israel, kazhdan@math.huji.ac.il

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## 2. GENERALIZED KAC-MOODY ALGEBRAS

Throughout the paper,  $k$  denotes a field of characteristic zero. All vector spaces in this paper will be over  $k$ .

**2.1.** We recall definitions from [K]. Let  $A = (a_{ij})$  be an  $n$ -by- $n$  matrix with entries in  $k$ , and  $(\mathfrak{h}, \Pi, \Pi^\vee)$  be a realization of  $A$ . This means that  $\mathfrak{h}$  is a vector space of dimension  $2n - \text{rank}(A)$ ,  $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$ ,  $\Pi^\vee = \{h_1, \dots, h_n\} \subset \mathfrak{h}$  are linearly independent, and  $\alpha_i(h_j) = a_{ji}$ .

**Definition.** The Lie algebra  $\tilde{\mathfrak{g}}(A)$  is generated by  $\mathfrak{h}, e_1, \dots, e_n, f_1, \dots, f_n$  with defining relations

$$[h, h'] = 0, \quad h, h' \in \mathfrak{h}; \quad [h, e_i] = \alpha_i(h)e_i; \quad [h, f_i] = -\alpha_i(h)f_i; \quad [e_i, f_j] = \delta_{ij}h_i.$$

We will denote  $\tilde{\mathfrak{g}}(A)$  simply by  $\tilde{\mathfrak{g}}$ , assuming that  $A$  has been fixed.

Let  $I$  be the sum of all two-sided ideals in  $\tilde{\mathfrak{g}}(A)$  which have zero intersection with  $\mathfrak{h} \subset \tilde{\mathfrak{g}}(A)$ . Let  $\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A)/I$ . The algebra  $\mathfrak{g}(A)$  is called a generalized Kac-Moody algebra. We will denote  $\mathfrak{g}(A)$  by  $\mathfrak{g}$ , assuming that  $A$  has been fixed. The Lie algebra  $\mathfrak{g}$  is graded by principal gradation ( $\deg(e_i) = 1, \deg(f_i) = -1, \deg(\mathfrak{h}) = 0$ ), and the homogeneous components are finite dimensional.

In the following we will assume that the matrix  $A$  is symmetrizable, i.e. there exists a collection of nonzero numbers  $d_i, i = 1, \dots, n$ , such that  $d_i a_{ij} = d_j a_{ji}$ . We will choose such a collection of numbers. Let us choose a nondegenerate bilinear symmetric form on  $\mathfrak{h}$  such that  $(h, h_i) = d_i^{-1} \alpha_i(h)$ . It is easy to see that such a form always exists. It is known [K] that there exists a unique extension of the form  $(\cdot, \cdot)$  to an invariant symmetric bilinear form  $(\cdot, \cdot)$  on  $\tilde{\mathfrak{g}}$ . (For this extension, one has  $(e_i, f_j) = \delta_{ij} d_i^{-1}$ ). The kernel of this form is  $I$ , and thus the form descends to a nondegenerate form on  $\mathfrak{g}$ .

**Remark.** One can show that forms on  $\mathfrak{g}$  coming from different forms on  $\mathfrak{h}$  are equivalent under automorphisms of  $\mathfrak{g}$ .

**2.2.** Let  $\mathfrak{n}_+, \mathfrak{n}_-, \mathfrak{b}_+, \mathfrak{b}_-$  be the nilpotent and the Borel subalgebras of  $\mathfrak{g}$  ( $\mathfrak{n}_+, \mathfrak{n}_-$  are generated by  $e_i$  and by  $f_i$ , respectively, and  $\mathfrak{b}_\pm := \mathfrak{n}_\pm \oplus \mathfrak{h}$ ). Let us regard  $\mathfrak{b}_+$  and  $\mathfrak{b}_-$  as Lie subalgebras of  $\mathfrak{g} \oplus \mathfrak{h}$  using the embeddings  $\eta_\pm : \mathfrak{b}_\pm \rightarrow \mathfrak{g} \oplus \mathfrak{h}$  given by

$$\eta_\pm(x) = x \oplus (\pm \bar{x}),$$

where  $\bar{x}$  is the image of  $x$  in  $\mathfrak{h}$ .

Define the inner product on  $\mathfrak{g} \oplus \mathfrak{h}$  by  $(\cdot, \cdot)_{\mathfrak{g} \oplus \mathfrak{h}} = (\cdot, \cdot)_{\mathfrak{g}} - (\cdot, \cdot)_{\mathfrak{h}}$ . The following proposition is well known and straightforward to check.

**Proposition 2.1.** The triple  $(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{b}_+, \mathfrak{b}_-)$  with the inner product  $(\cdot, \cdot)_{\mathfrak{g} \oplus \mathfrak{h}}$  and embeddings  $\eta_\pm$  is a (graded) Manin triple.

The proposition implies that  $\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{b}_+, \mathfrak{b}_-$  are naturally Lie bialgebras, with  $\mathfrak{b}_+^* = \mathfrak{b}_-^{op}$ , where  $*$  denotes the restricted dual space, and  $^{op}$  denotes the opposite cocommutator. The cocommutator  $\delta$  on these algebras is easily computed:

$$(2.1) \quad \delta(h) = 0, h \in \mathfrak{h} \subset \mathfrak{b}_\pm; \quad \delta(e_i) = \frac{1}{2} d_i e_i \wedge h_i; \quad \delta(f_i) = \frac{1}{2} d_i f_i \wedge h_i.$$

The Lie subalgebra  $\{(0, h) | h \in \mathfrak{h}\}$  is thus an ideal and coideal in  $\mathfrak{g} \oplus \mathfrak{h}$ , and so the quotient  $\mathfrak{g} = (\mathfrak{g} \oplus \mathfrak{h})/\mathfrak{h}$  is also a Lie bialgebra with Lie subbialgebras  $\mathfrak{b}_+, \mathfrak{b}_-$ , and the same cocommutator formulas.

In fact, the same formulas define a Lie bialgebra structure on  $\tilde{\mathfrak{g}}$  and its Borel subalgebras  $\tilde{\mathfrak{b}}_{\pm}$  (generated by  $\mathfrak{h}, e_i$  and  $\mathfrak{h}, f_i$ , respectively). The projections  $\tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ ,  $\tilde{\mathfrak{b}}_{\pm} \rightarrow \mathfrak{b}_{\pm}$  are thus Lie bialgebra homomorphisms.

**Remark.** The factors  $\frac{1}{2}$  in (2.1) appear because for  $a \in \mathfrak{h} \subset \mathfrak{b}_+$  and  $b \in \mathfrak{h} \subset \mathfrak{b}_-$ , one has  $(a, b)_{\mathfrak{g} \oplus \mathfrak{h}} = (a + \bar{a}, b - \bar{b}) = 2(a, b)$ .

### 3. QUANTIZATION OF GENERALIZED KAC-MOODY ALGEBRAS

**3.1.** Let  $\hbar$  be a formal parameter, and  $q = e^{\hbar/2}$ . Let  $\Phi = 1 + \frac{\hbar^2}{24}[\Omega_{12}, \Omega_{23}] + \dots$  be a universal Lie associator (see [Dr4]), and  $\mathfrak{a} \rightarrow U_{\hbar}(\mathfrak{a})$  be the functor of quantization of Lie bialgebras associated with  $\Phi$  (see [EK1, EK2]). In this section we will describe explicitly  $U_{\hbar}(\mathfrak{a})$ , when  $\mathfrak{a}$  is one of the Lie bialgebras of the previous section.

**Proposition 3.1.** *The QUE algebra  $U_{\hbar}(\tilde{\mathfrak{b}}_+)$  is isomorphic to the QUE algebra  $\tilde{\mathcal{U}}_+$  generated (topologically) by  $\mathfrak{h}$  and elements  $E_i$ ,  $i = 1, \dots, n$ , with the relations*

$$[h, h'] = 0; [h, E_i] = \alpha_i(h)E_i, h, h' \in \mathfrak{h},$$

with coproduct

$$\Delta(h) = h \otimes 1 + 1 \otimes h; \Delta(E_i) = E_i \otimes q^{\gamma_i} + 1 \otimes E_i,$$

for suitable elements  $\gamma_i \in \mathfrak{h}[[\hbar]]$ .

*Proof.* Since  $U_{\hbar}$  is a functor, the embedding of Lie bialgebras  $\mathfrak{h} \rightarrow \tilde{\mathfrak{b}}_+$  defines an embedding of QUE algebras  $U_{\hbar}(\mathfrak{h}) = U(\mathfrak{h})[[\hbar]] \rightarrow U_{\hbar}(\tilde{\mathfrak{b}}_+)$ .

Also,  $\mathfrak{b}_+$  has a  $\mathbb{Z}_+^n$ -grading given by  $\deg_i(h) = 0$ ,  $\deg_i(e_j) = \delta_{ij}$ , so by functoriality the quantized algebra  $U_{\hbar}(\mathfrak{b}_+)$  has a grading by  $\mathbb{Z}_+^n$  as well (as this grading is simply an action of  $\mathbb{G}_m^n$ ).

As a result, we get  $U_{\hbar}(\tilde{\mathfrak{b}}_+) = \bigoplus_{\mathbf{m} \in \mathbb{Z}_+^n} U_{\hbar}(\tilde{\mathfrak{b}}_+)[\mathbf{m}]$ , where  $U_{\hbar}(\tilde{\mathfrak{b}}_+)[\mathbf{m}]$  are free modules over  $U_{\hbar}(\mathfrak{h})$  of finite rank (in fact, the same rank as before deformation). In particular, if  $\mathbf{m} = 1_j$ , where  $1_j(i) = \delta_{ij}$ , then  $U_{\hbar}(\tilde{\mathfrak{b}}_+)[\mathbf{m}]$  has rank 1.

Let us choose an element  $E'_j$  in  $U_{\hbar}(\tilde{\mathfrak{b}}_+)[1_j]$  which equals  $e_j$  modulo  $\hbar$ .

For homogeneity reasons we have

$$\Delta(E'_j) = (E'_j \otimes 1)\Psi_1 + (1 \otimes E'_j)\Psi_2,$$

where  $\Psi_i \in 1 + \hbar U(\mathfrak{h} \oplus \mathfrak{h})[[\hbar]]$ .

We have  $(\Delta \otimes 1)(\Delta(E'_i)) = (1 \otimes \Delta)(\Delta(E'_i))$ . This implies the following equations on  $\Psi_1, \Psi_2$ :

$$(3.1) \quad (\Psi_1 \otimes 1)(\Delta \otimes 1)(\Psi_1) = (1 \otimes \Delta)(\Psi_1),$$

$$(3.2) \quad (\Psi_2 \otimes 1)(\Delta \otimes 1)(\Psi_1) = (1 \otimes \Psi_1)(1 \otimes \Delta)(\Psi_2),$$

$$(3.3) \quad (1 \otimes \Psi_2)(1 \otimes \Delta)(\Psi_2) = (\Delta \otimes 1)(\Psi_2).$$

Let us regard  $\Psi_i$  as functions of two variables  $x, y \in \mathfrak{h}^*$ , and let  $\psi_i = \log \Psi_i$ . Then (3.1)-(3.3) can be written in the form

$$(3.4) \quad \psi_1(x, y) + \psi_1(x + y, z) = \psi_1(x, y + z),$$

$$(3.5) \quad \psi_2(y, z) + \psi_2(x, y + z) = \psi_2(x + y, z),$$

$$(3.6) \quad \psi_1(y, z) + \psi_2(x, y + z) = \psi_1(x + y, z) + \psi_2(x, y).$$

Let us set  $z = -x - y$  in (3.4). We get

$$(3.7) \quad \psi_1(x, y) = \phi_1(x) - \phi_1(x + y),$$

where  $\phi_1(x) = \psi_1(x, -x)$ . Similarly, from equation (3.6), putting  $x = -y - z$ , we get

$$(3.8) \quad \psi_2(y, z) = \phi_2(z) - \phi_2(y + z),$$

where  $\phi_2(z) = \psi_2(-z, z)$ .

It is easy to check that after these substitutions, equation (3.5) becomes

$$(3.9) \quad \phi_1(y) - \phi_1(x + y) - \phi_1(y + z) + \phi_1(x + y + z) = \phi_2(y) - \phi_2(x + y) - \phi_2(y + z) + \phi_2(x + y + z).$$

Let  $\gamma(x) = \frac{1}{\log q}(\phi_1(x) - \phi_2(x))$ . We have

$$(3.10) \quad \gamma(y) - \gamma(x + y) - \gamma(y + z) + \gamma(x + y + z) = 0.$$

In particular,  $d^2\gamma = 0$  and hence  $\gamma$  is an affine linear function (i.e.  $\gamma \in (\mathfrak{h} \oplus k)[[\hbar]]$ ). We will denote  $\gamma$  by  $\gamma_i$  to remember its dependence on  $i$ .

Define  $E_i = E'_i e^{-\phi_2(x)}$ . Then it is easy to see from the above that

$$(3.11) \quad \Delta(E_i) = E_i \otimes q^{\gamma_i} + 1 \otimes E_i.$$

From this we see using the counit axiom that the constant terms of  $\gamma_i$  are zero, i.e.  $\gamma_i$  are elements of  $\mathfrak{h}[[\hbar]]$ .

It is also clear that  $\mathfrak{h}$  and  $E_i$  topologically generate  $U_{\hbar}(\tilde{\mathfrak{b}}_+)$  and that the only relations are the ones given in the theorem (this follows from the fact that  $\tilde{\mathfrak{n}}_+$  is a free Lie algebra). The proposition is proved.  $\square$

**3.2.** Let us now compute the elements  $\gamma_i$ .

**Proposition 3.2.** *One has  $\gamma_i = d_i h_i$ .*

*Proof.* We have a surjective map of Lie bialgebras  $\tilde{\mathfrak{b}}_+ \rightarrow \mathfrak{b}_+$ . By functoriality of quantization, this defines a surjective homomorphism of QUE algebras  $U_{\hbar}(\tilde{\mathfrak{b}}_+) \rightarrow U_{\hbar}(\mathfrak{b}_+)$  (which preserves the grading). Therefore,  $U_{\hbar}(\mathfrak{b}_+)$  is also generated by  $\mathfrak{h}, E_i$  satisfying the relations

$$[h, E_i] = \alpha_i(h) E_i$$

(and maybe some additional relations), and the coproduct is defined by

$$\Delta(h) = 0, \Delta(E_i) = E_i \otimes q^{\gamma_i} + 1 \otimes E_i.$$

It follows from the definition of the Lie bialgebra  $\mathfrak{b}_+$  that it is self-dual in the graded sense:  $\mathfrak{b}_+ \cong \mathfrak{b}_+^*$ . Thus,  $U_{\hbar}(\mathfrak{b}_+) \cong U_{\hbar}(\mathfrak{b}_+^*)$ . By the result of [EG], we have  $U_{\hbar}(\mathfrak{b}_+^*) \cong U_{\hbar}(\mathfrak{b}_+^{op})^{*op}$ . On the other hand, for any Lie bialgebra  $\mathfrak{a}$ ,  $U_{\hbar}(\mathfrak{a}^{op}) \cong U_{-\hbar}(\mathfrak{a})$  (since the universal quantization formulas of [EK2] are written in terms of  $[\cdot, \cdot]$  and  $\hbar\delta$ ). Therefore,  $U_{\hbar}(\mathfrak{b}_+^{op})^{*op} \cong U_{-\hbar}(\mathfrak{b}_+)^{*op}$ . Thus, we have an (graded) isomorphism of QUE algebras  $U_{\hbar}(\mathfrak{b}_+) \rightarrow U_{-\hbar}(\mathfrak{b}_+)^{*op}$ , which in degree zero comes from the identification  $\mathfrak{h} \rightarrow \mathfrak{h}^*$  using the form  $2(\cdot, \cdot)$  on  $\mathfrak{h}$ . This isomorphism can be understood as a bilinear form  $B : U_{\hbar}(\mathfrak{b}_+) \otimes U_{-\hbar}(\mathfrak{b}_+) \rightarrow k((\hbar))$  satisfying the conditions

$$B(xy, z) = B(y \otimes x, \Delta(z)), \quad B(z, xy) = B(\Delta(z), x \otimes y),$$

such that

$$B(q^a, q^b) = q^{-(a,b)}, \quad a, b \in \mathfrak{h}.$$

Let  $B_i = B(E_i, E_i)$ ; clearly, this is nonzero. Using the properties of  $B$ , we have

$$B(E_i, E_i q^a) = q^{-(a, \gamma_i)} B_i; \quad B(E_i, q^a E_i) = B_i.$$

But  $q^a E_i q^{-a} = q^{\alpha_i(a)} E_i$ , so we get  $(a, \gamma_i) = \alpha_i(a)$ , which yields  $\gamma_i = d_i h_i$ , as desired.  $\square$

Thus we have proved

**Theorem 3.3.** *The QUE algebra  $U_{\hbar}(\tilde{\mathfrak{b}}_+)$  is isomorphic to the QUE algebra  $\tilde{\mathcal{U}}_+$  generated (topologically) by  $\mathfrak{h}$  and elements  $E_i$ ,  $i = 1, \dots, n$ , with the relations*

$$[h, h'] = 0, \quad [h, E_i] = \alpha_i(h) E_i, \quad h, h' \in \mathfrak{h},$$

with coproduct

$$\Delta(h) = 0, \Delta(E_i) = E_i \otimes q^{d_i h_i} + 1 \otimes E_i.$$

**3.3.** Now let us describe explicitly the QUE algebra  $U_{\hbar}(\mathfrak{b}_+)$ .

**Theorem 3.4.** *There exists a unique symmetric bilinear form  $B$  on  $U_{\hbar}(\tilde{\mathfrak{b}}_+)$  with values in  $k((\hbar))$  which satisfies the properties*

$$B(xy, z) = B(x \otimes y, \Delta(z)), \quad B(z, xy) = B(\Delta(z), x \otimes y),$$

$$B(q^a, q^b) = q^{-(a,b)}, \quad a, b \in \mathfrak{h},$$

$$B(E_i, E_j) = \frac{\delta_{ij}}{q - q^{-1}}.$$

The QUE algebra  $U_{\hbar}(\mathfrak{b}_+)$  is isomorphic to the quotient  $\mathcal{U}_+$  of  $\tilde{\mathcal{U}}_+$  (as in Theorem 3.3) by the Hopf ideal  $\text{Ker}(B)$ .

*Proof.* The existence and uniqueness of  $B$  easily follows from the freeness of the algebra generated by  $E_i$ . Also, the uniqueness of  $B$  implies that  $B$  is symmetric.

Moreover, we claim that a form with the same properties exists and is unique on the quotient algebra  $U_{\hbar}(\mathfrak{b}_+)$ . Indeed, we have seen that  $U_{\hbar}(\mathfrak{b}_+) \cong U_{-\hbar}(\mathfrak{b}_+)^{*op}$ . On the other hand, by using the conjugation by  $q^{-\sum x_i^2/2}$ , where  $x_i$  is an orthonormal basis of  $\mathfrak{h}$ , we find that  $U_{-\hbar}(\mathfrak{b}_+)^{op} \cong U_{\hbar}(\mathfrak{b}_+)$ . This yields a symmetric isomorphism  $U_{\hbar}(\mathfrak{b}_+) \rightarrow U_{\hbar}(\mathfrak{b}_+)^*$ , which gives a desired form. (Here we use that for any Hopf algebra  $H$ , the Hopf algebra  $H^{*op}$  is isomorphic to  $H^{op*}$  via the antipode.)

Thus, the form on the big algebra is pulled back from the form on the small algebra, and hence the kernel of the projection from the big algebra to the small one is contained in  $Ker(B)$ .

Finally, it is clear that the form  $B$  on  $U_{\hbar}(\mathfrak{b}_+)$  is nondegenerate (as it corresponds to an isomorphism). So  $U_{\hbar}(\mathfrak{b}_+) = \tilde{\mathcal{U}}_+/Ker(B) = \mathcal{U}_+$ , as desired.  $\square$

**Corollary 3.5.**  $\mathcal{U}_+$  is a flat deformation of  $U(\mathfrak{b}_+)$ .

*Proof.* Clear, as  $U_{\hbar}(\mathfrak{b}_+)$  is flat by definition.  $\square$

**Corollary 3.6.** Suppose that  $A$  is a generalized Cartan matrix (i.e.  $a_{ii} = 2$ , and  $a_{ij}$  are nonpositive integers for  $i \neq j$ ). In this case, the two-sided ideal  $Ker(B)$  is generated by the quantum Serre relations

$$\sum_{m=0}^{1-a_{ij}} \frac{(-1)^m}{[m]_{q_i}! [1-a_{ij}-m]_{q_i}!} E_i^{1-a_{ij}-m} E_j E_i^m = 0,$$

where  $q_i = q^{d_i}$ .

*Proof.* It is known ([L], Section 1) that the ideal generated by the quantum Serre relations is contained in  $Ker(B)$ . Besides, we know [K] that  $\mathfrak{b}_+$  is the quotient of  $\tilde{\mathfrak{b}}_+$  by the classical limits of the Serre relations. This fact and Corollary 3.5 imply the result.  $\square$

**Theorem 3.7.** The QUE algebra  $U_{\hbar}(\mathfrak{g})$  is isomorphic to the quotient  $\mathcal{U}$  of the (restricted) quantum double  $D(\mathcal{U}_+)$  by the ideal generated by the identification of  $\mathfrak{h} \subset \mathcal{U}_+$  and  $\mathfrak{h}^* \subset \mathcal{U}_+^*$ . In particular, if  $A$  is a generalized Cartan matrix then  $U_{\hbar}(\mathfrak{g})$  is isomorphic to the Drinfeld-Jimbo quantum group associated to the Kac-Moody algebra  $\mathfrak{g}$  (see [Dr2], Example 6.2, and [J]).

**Remark.** The word “restricted” means that as a  $k[[\hbar]]$ -module,  $D(\mathcal{U}_+) = \mathcal{U}_+ \otimes \mathcal{U}_+^*$ , where  $\mathcal{U}_+^*$  is the restricted (by the grading) dual space to  $\mathcal{U}_+$ .

Thus, Theorem 3.7 constructs a flat deformation of  $U(\mathfrak{g})$ , and thus answers Question 8.3 from [Dr1].

*Proof.* This follows from the previous results and the fact that quantization commutes with taking the double, see [EG].  $\square$

Now define  $\tilde{\mathfrak{g}}'$  to be the (restricted) Drinfeld double of  $\tilde{\mathfrak{b}}_+$ , as a Lie bialgebra.

**Remark.** We note that while for generic  $A$  we have  $\tilde{\mathfrak{g}} \cong \tilde{\mathfrak{g}}'$  as graded Lie algebras, for special values of  $A$  this is not the case, and in particular the Lie algebra  $\tilde{\mathfrak{g}}'$  is not generated by elements of degree 1 and  $-1$ .

**Theorem 3.8.** The QUE algebra  $U_{\hbar}(\tilde{\mathfrak{g}}')$  is isomorphic to the quotient  $\tilde{\mathcal{U}}'$  of the (restricted) quantum double  $D(\tilde{\mathcal{U}}_+)$  by the ideal generated by the identification of  $\mathfrak{h} \subset \tilde{\mathcal{U}}_+$  and  $\mathfrak{h}^* \subset \tilde{\mathcal{U}}_+^*$ .

*Proof.* The proof of this theorem is the same as the proof of Theorem 3.7.  $\square$

#### 4. CATEGORY $\mathcal{O}$

**4.1.** Let  $\mathfrak{g}_+$  be a Lie bialgebra, and  $U_{\hbar}(\mathfrak{g}_+)$  be its quantization as in [EK2]. Then one can define the standard notion of a Drinfeld-Yetter module, or a dimodule, over  $\mathfrak{g}_+$  and  $U_{\hbar}(\mathfrak{g}_+)$  (see e.g. [EK2]). Let  $\mathcal{M}$  be the category of deformation dimodules over  $\mathfrak{g}_+$ , i.e. of  $\mathfrak{g}_+$ -dimodules realized on a topologically free  $k[[\hbar]]$ -module. Let  $\mathcal{M}_{\hbar}$  be the category of dimodules over  $U_{\hbar}(\mathfrak{g}_+)$ . Recall that both categories are braided tensor categories: the category  $\mathcal{M}$  has the braided tensor structure defined by the associator  $\Phi$ , while the category  $\mathcal{M}_{\hbar}$  has the braided tensor structure obtained from the “universal R-matrix” (see [EK2]).

**Theorem 4.1.** *There exists an equivalence of braided tensor categories  $\mathcal{M} \rightarrow \mathcal{M}_{\hbar}$ , which is the identity functor at the level of  $k[[\hbar]]$ -modules (i.e., there exists a consistent system of isomorphisms of  $k[[\hbar]]$ -modules,  $V \rightarrow F(V)$ ).*

*Proof.* We will use the notation of [EK1,EK2]. Recall that in [EK1,EK2], we defined the functor  $F$  from the category of deformation  $\mathfrak{g}_+$ -dimodules to the category of  $k[[\hbar]]$ -modules by  $F(V) := \text{Hom}(M_-, M_+^* \hat{\otimes} V)$ , and equipped it with a tensor structure (here  $\hat{\otimes}$  is the completed tensor product with respect to the weak topology in  $M_+^*$ ). To turn  $F$  into a functor we are looking for, we need to introduce on  $F(V)$ , for all  $V$ , an action and a coaction of  $U_{\hbar}(\mathfrak{g}_+)$ .

Recall that  $U_{\hbar}(\mathfrak{g}_+)$  is defined in [EK1,EK2] to be the space  $F(M_-)$ . The action of  $U_{\hbar}(\mathfrak{g}_+)$  on  $F(V)$  is explicitly defined in [EK1], section 9. Namely, if  $v \in F(V)$  and  $a \in U_{\hbar}(\mathfrak{g}_+) = F(M_-)$ , one defines  $av \in F(V)$  to be  $(i_+^* \otimes 1 \otimes 1) \circ (1 \otimes v) \circ a$ .

Now define the coaction of  $U_{\hbar}(\mathfrak{g}_+)$  on  $F(V)$ . Since  $F$  is a tensor functor, the braiding map for  $\mathfrak{g}_+$ -dimodules composed with the permutation of components defines a map  $R : F(M_-) \otimes F(V) \rightarrow F(M_-) \otimes F(V)$  (the universal R-matrix). The coaction of  $U_{\hbar}(\mathfrak{g}_+)$  is defined by the map  $v \rightarrow R(1 \otimes v)$ , where 1 is the unit of  $U_{\hbar}(\mathfrak{g}_+)$ . One can check that these action and coaction are compatible, so they define a structure of a  $U_{\hbar}(\mathfrak{g}_+)$ -dimodule on  $F(V)$ . Thus  $F$  becomes a functor from  $\mathfrak{g}_+$ -dimodules to  $U_{\hbar}(\mathfrak{g}_+)$ -dimodules. It is straightforward to check that this functor equipped with the tensor structure of [EK1] is a braided tensor functor between these categories.

It remains to show that  $F$  is an equivalence of categories. To do this, it is sufficient to construct the inverse functor. This is done using twisting the tensor category of dimodules of  $U_{\hbar}(\mathfrak{g}_+)$  by a family  $a(t)$  of elements of the Grothendieck-Teichmüller group, as explained in Section 2 of [EK2]. The theorem is proved.  $\square$

**Remark.** We use this opportunity to correct the formulation of Theorem 6.2 in [EK1], whose original formulation is not quite correct. Instead of the category  $\mathcal{M}_{\mathfrak{a}}$  of  $\mathfrak{a}$ -modules, considered in this theorem, one should consider the category  $\tilde{\mathcal{M}}_{\mathfrak{a}}$  of deformation  $\mathfrak{a}$ -modules. The functor  $F$  in the theorem (from  $\mathcal{M}_{\mathfrak{a}}$  to the category  $\mathcal{R}$  of representations of  $U_{\hbar}(\mathfrak{a})$ ) naturally extends to  $\tilde{\mathcal{M}}_{\mathfrak{a}}$ . The correct formulation of Theorem 6.2 says that  $F$  is an equivalence of  $\tilde{\mathcal{M}}_{\mathfrak{a}}$  onto  $\mathcal{R}$  (the proof of this is obvious from the results of [EK1]). In this form, Theorem 6.2 of [EK1] (for  $\mathfrak{a}$  being the double of a finite dimensional Lie bialgebra) is a special case of Theorem 4.1 above.

**4.2.** Let us return to the setting of generalized Kac-Moody algebras. Recall that the category  $\mathcal{O}$  for  $\mathfrak{g}$  is defined to be the category of  $\mathfrak{h}$ -diagonalizable  $\mathfrak{g}$ -representations, whose weights belong to a union of finitely many cones  $\lambda - \sum_i \mathbb{Z}_+ \alpha_i$ ,

$\lambda \in \mathfrak{h}^*$ , and the weight subspaces are finite dimensional. Define also the category  $\mathcal{O}[[\hbar]]$  of deformation representations of  $\mathfrak{g}$ , i.e. representations of  $\mathfrak{g}$  on topologically free  $k[[\hbar]]$ -modules with the above properties (with  $\lambda \in \mathfrak{h}^*[[\hbar]]$ ).

In a similar way one defines the category  $\mathcal{O}_{\hbar}$  for the algebra  $\mathcal{U}$ : it is the category of  $\mathcal{U}$ -modules which are topologically free over  $k[[\hbar]]$  and satisfy the same conditions as in the classical case.

Let  $\Omega \in \mathfrak{g} \hat{\otimes} \mathfrak{g}$  (where  $\hat{\otimes}$  is the tensor product completed with respect to the grading) be the inverse element to the bilinear form  $(,)$  on  $\mathfrak{g}$ . It defines an operator in any tensor product  $V \otimes W$  of modules from category  $\mathcal{O}[[\hbar]]$ . Following Drinfeld, we put on  $\mathcal{O}[[\hbar]]$  a structure of a braided tensor category using the associator  $\Phi$ , with braiding  $q^{\Omega}$  (see [EK1]). The category  $\mathcal{O}_{\hbar}$  is also a braided tensor category, with braiding defined by the universal R-matrix  $R \in U_{\hbar}(\mathfrak{b}_+) \hat{\otimes} U_{\hbar}(\mathfrak{b}_-)$  coming from the isomorphism  $U_{\hbar}(\mathfrak{b}_+) \rightarrow U_{\hbar}(\mathfrak{b}_-)^{*op}$ .

Recall that a highest weight module over  $\mathfrak{g}$  or  $U_{\hbar}(\mathfrak{g})$  is a module generated by a highest weight vector, and that for each highest weight  $\lambda$ , we have the Verma module  $M(\lambda)$  and the irreducible module  $L(\lambda)$ , and any highest weight module  $N$  with highest weight  $\lambda$  can be included in a diagram  $M(\lambda) \rightarrow N \rightarrow L(\lambda)$ , where both maps are surjective, and defined uniquely up to scaling.

**Theorem 4.2.** *There exists an equivalence of braided tensor categories  $F : \mathcal{O}[[\hbar]] \rightarrow \mathcal{O}_{\hbar}$ , which is isomorphic to the identity functor at the level of  $\mathfrak{h}$ -graded  $k[[\hbar]]$ -modules. This equivalence maps the Verma (resp. irreducible) module with highest weight  $\lambda$  to the Verma (resp. irreducible) module with highest weight  $\lambda$ .*

*Proof.* First of all, by Theorem 3.4, we can replace  $\mathcal{U}$  with  $D(U_{\hbar}(\mathfrak{b}_+))/(\mathfrak{h} = \mathfrak{h}^*)$ .

Now, in order to construct the functor  $F$ , it is enough to construct a similar functor between the corresponding categories for the algebras  $D(\mathfrak{b}_+)$  and  $D(U_{\hbar}(\mathfrak{b}_+))$ , i.e. between certain categories of dimodules over  $\mathfrak{b}_-$  and  $U_{\hbar}(\mathfrak{b}_-)$ .

But such a functor was constructed in Theorem 4.1. Indeed, since all our constructions are compatible with the weight decompositions, the functor of Theorem 4.1 restricts to an equivalence  $F$  between the categories  $\mathcal{O}[[\hbar]]$  and  $\mathcal{O}_{\hbar}$ .

The second statement is obvious from the construction. Namely, it is easy to see that any formal deformation of a highest weight  $\mathfrak{g}$ -module to a  $U_{\hbar}(\mathfrak{g})$ -module that has the same character is necessarily a highest weight module. This fact and the compatibility of  $F$  with the weight decomposition imply that under  $F$ , a highest weight module goes to a highest weight module, a Verma module to a Verma module, and an irreducible module to an irreducible module (with the same highest weight). Indeed, the first statement follows from the fact that any highest weight module with character equal to the character of  $M(\lambda)$  is isomorphic to  $M(\lambda)$ , and the second one from the fact that a formal deformation of an irreducible module is irreducible. The theorem is proved.  $\square$

**Corollary 4.3.** *The characters of irreducible highest weight modules over  $\mathcal{U}$  are the same as those for  $U(\mathfrak{g})$ .*

Corollary 4.3 answers positively question 8.1 from [Dr1].

**Remark.** In fact, it is easy to see that Theorem 4.2 implies a positive answer to Question 8.2 of [Dr1]. Namely, following [Dr1], define  $I_{\beta}$  to be the left ideal in  $U(\mathfrak{g})$  generated by elements of weight  $\leq \beta$ . We can define a similar ideal  $I_{\beta}^{\hbar}$  in  $U_{\hbar}(\mathfrak{g})$ . Then the modules  $U(\mathfrak{g})/I_{\beta}$ ,  $U_{\hbar}(\mathfrak{g})/I_{\beta}^{\hbar}$  are in the categories  $\mathcal{O}$ ,  $\mathcal{O}_{\hbar}$ , and we



have  $F(U(\mathfrak{g})/I_\beta) = U_{\hbar}(\mathfrak{g})/I_\beta^{\hbar}$ , by a deformation argument. (Indeed, the module  $U(\mathfrak{g})/I_\beta$  is generated by a vector  $v$  with the only relation  $I_\beta v = 0$ , and  $F(U(\mathfrak{g})/I_\beta)$  is a deformation of  $U(\mathfrak{g})/I_\beta$  which has a vector killed by  $I_\beta^{\hbar}$ , because weights are preserved by our construction, so it's  $U_{\hbar}(\mathfrak{g})/I_\beta^{\hbar}$ ). Thus, we have natural isomorphisms  $\psi_\beta : \text{End}_{U(\mathfrak{g})}(U(\mathfrak{g})/I_\beta)[[\hbar]] \rightarrow \text{End}_{U_{\hbar}(\mathfrak{g})}(U_{\hbar}(\mathfrak{g})/I_\beta^{\hbar})$ . Taking the inverse limit of  $\psi_\beta$  with respect to  $\beta$  (as the multiplicities of simple roots in  $\beta$  go to  $+\infty$ ), we get an isomorphism of algebras  $\psi : \hat{U}(\mathfrak{g})[[\hbar]] \rightarrow \hat{U}_{\hbar}(\mathfrak{g})$ , required in Question 8.2 of [Dr1].

**Corollary 4.4.** *(The Drinfeld-Kohno theorem for  $\mathfrak{g}$ ). Let  $k = \mathbb{C}$ . Let  $V \in \mathcal{O}[[\hbar]]$ , and  $V_q = F(V)$  be its image in  $\mathcal{O}_{\hbar}$ . Consider the system of the Knizhnik-Zamolodchikov differential equations with respect to a function  $\mathcal{F}(z_1, \dots, z_n)$  of complex variables  $z_1, \dots, z_n$  with values in  $V^{\otimes n}[\lambda][[\hbar]]$  (the weight subspace of weight  $\lambda$ ):*

$$\frac{\partial \mathcal{F}}{\partial z_i} = \frac{\hbar}{2\pi i} \sum_{j \neq i} \frac{\Omega_{ij} \mathcal{F}}{z_i - z_j}.$$

*Then the monodromy representation of the braid group  $B_n$  for this equation is isomorphic to the representation of  $B_n$  on  $V_q^{\otimes n}[\lambda]$  defined by the formula*

$$b_i \rightarrow \sigma_i R_{ii+1},$$

*where  $b_i$  are generators of the braid group and  $\sigma_i$  are the permutation of the  $i$ -th and  $(i+1)$ -th components.*

**Remark.** As usual, we identify  $\pi_1(\mathbb{C}^n \setminus \{z_i = z_j\}/S_n)$  with  $B_n$  by picking the reference point  $(1, 2, \dots, n) \in \mathbb{C}^n$ .

*Proof.* The result follows directly from Theorem 4.2 if we take  $\Phi$  to be the Knizhnik-Zamolodchikov associator: in this case, the two representations are the braid group actions on the  $n$ -th power of two objects in  $\mathcal{O}$ ,  $\mathcal{O}_{\hbar}$ , which correspond to each other under the braided tensor equivalence  $F$ .  $\square$

It is easy to generalize these results to the algebra  $\tilde{\mathfrak{g}}'$ . Namely, define the categories  $\tilde{\mathcal{O}}[[\hbar]]$  and  $\tilde{\mathcal{O}}$  of representations of  $\tilde{\mathfrak{g}}'$ ,  $\mathcal{U}'$  similarly to the definition  $\mathcal{O}[[\hbar]]$ ,  $\mathcal{O}_{\hbar}$ . These categories are braided in a similar way to  $\mathcal{O}[[\hbar]]$ ,  $\mathcal{O}_{\hbar}$ , and we have

**Theorem 4.5.** *There exists an equivalence of braided tensor categories  $F : \tilde{\mathcal{O}}[[\hbar]] \rightarrow \tilde{\mathcal{O}}_{\hbar}$ , which is isomorphic to the identity functor at the level of  $\mathfrak{h}$ -graded  $k[[\hbar]]$ -modules. This equivalence maps the Verma (resp. irreducible) module with highest weight  $\lambda$  to the Verma (resp. irreducible) module with highest weight  $\lambda$ .*

*Proof.* The proof is the same as the proof of Theorem 4.2, using Theorem 3.8.  $\square$

**Remark.** Note that in this theorem, we could not use  $\tilde{\mathfrak{g}}$  instead of  $\tilde{\mathfrak{g}}'$ , since  $\tilde{\mathfrak{g}}$ , in general, does not admit a nondegenerate invariant form, and thus one cannot define the element  $\Omega$  which is necessary to define the tensor structure.

**Corollary 4.6.** *The obvious analog of Corollary 4.4 is valid if  $\mathfrak{g}$  is replaced with  $\tilde{\mathfrak{g}}'$ .*

**Remark.** We note that Corollary 4.4 for irreducible integrable modules and Corollary 4.6 for Verma and contragredient Verma modules were proved in [V].

**4.3.** Now we want to formulate analytic versions of the results of the previous subsection in which  $k = \mathbb{C}$  and  $\hbar$  is no longer a formal parameter but a complex number. We give such versions in this subsection. We note that for reader's convenience we do not state our results in the maximal possible generality.

Let us assume for simplicity that the algebra  $\mathfrak{g}$  (in particular, the matrix  $A$ ) is defined over  $\mathbb{Q}$  (this is definitely the case for generalized Cartan matrices). Let  $\mathcal{O}_{\mathbb{Q}}$  be the full subcategory of the category  $\mathcal{O}$  for  $\mathfrak{g}$  consisting of modules whose weights are defined over  $\mathbb{Q}$ .

Let  $\hbar \in \mathbb{C}$ ,  $q = e^{\hbar/2}$ . By  $q^X$  we will always mean  $e^{\hbar X/2}$ . Let  $\mathcal{U}_{\hbar}$  be the Drinfeld-Jimbo quantum group, generated by  $E_i, F_i, q^h$ ,  $h \in \mathfrak{h}$ , with the usual relations, and the relations defined by the kernel of the bilinear form  $B$ . Let  $\mathcal{O}_{\mathbb{Q}, \hbar}$  be the full subcategory of the category  $\mathcal{O}$  for  $\mathcal{U}_{\hbar}$ , consisting of modules whose weights are defined over  $\mathbb{Q}$ .

For any  $\hbar \in \mathbb{C}$  which is not a nonzero rational multiple of  $\pi i$  (i.e. is such that  $q$  is not a nontrivial root of unity), one can define the structure of a tensor category on both  $\mathcal{O}_{\mathbb{Q}}$  and  $\mathcal{O}_{\mathbb{Q}, \hbar}$ .

Indeed, by standard facts about linear ordinary differential equations, the series in  $\hbar$  obtained by restricting the Knizhnik-Zamolodchikov associator  $\Phi$  to a weight subspace in the tensor product of three objects  $\mathcal{O}_{\mathbb{Q}}$  is convergent for small  $\hbar$ , and the resulting analytic function continues (in a single-valued fashion) to the values of  $\hbar$  not belonging to  $\pi i \mathbb{Q}$  (as for such  $\hbar$  the eigenvalues of the operator  $\Omega_{ij}$  never differ by a nonzero integer, i.e., no resonances occur). This allows us to define a tensor structure on  $\mathcal{O}_{\mathbb{Q}}$  (see also [KL]).

The structure of a tensor category on  $\mathcal{O}_{\mathbb{Q}, \hbar}$  comes from the Hopf algebra structure on  $\mathcal{U}_{\hbar}$ . Moreover, the first category is braided, with braiding  $e^{\hbar \Omega/2}$ , and the second category is braided with braiding defined by the  $R$ -matrix, which is well defined for generic  $\hbar$ , i.e., outside of a countable set (indeed, the  $R$ -matrix is inverse to the Drinfeld pairing, and this pairing is nondegenerate for formal  $\hbar$ , so has countably many zeros for numerical  $\hbar$ ).

**Theorem 4.7.** *If  $\hbar$  is generic (i.e. outside of a countable set), then there exists a braided tensor functor  $F_{\hbar} : \mathcal{O}_{\mathbb{Q}} \rightarrow \mathcal{O}_{\mathbb{Q}, \hbar}$ , which is the identity functor at the level of  $\mathfrak{h}$ -graded vector spaces, and maps Verma modules to Verma modules and irreducible modules to irreducible modules.*

*Proof.* The theorem is proved similarly to Theorem 4.2. Namely, consider the functor  $F$  constructed in Theorem 4.2. One can check directly that for any  $V \in \mathcal{O}_{\mathbb{Q}}$ , the structure maps for  $F(V)$  are defined by finite expressions of the associator  $\Phi$  and the braiding  $e^{\hbar \Omega/2}$ , which implies that they make sense for complex  $\hbar \notin \pi i \mathbb{Q}$ .  $\square$

**Remark.** In the case when  $\mathfrak{g}$  is a finite dimensional semisimple Lie algebra, the restriction of the functor  $F_{\hbar}$  to the category of finite dimensional modules is the functor constructed in [KL]. The general construction is, essentially, by analogy with [KL].

**Corollary 4.8.** *For generic  $\hbar$ , the character of the irreducible module  $L_{\hbar}(\lambda)$  over  $\mathcal{U}$  with highest weight  $\lambda \in \mathfrak{h}(\mathbb{Q})$  is the same as the character of the corresponding irreducible module  $L(\lambda)$  over  $\mathfrak{g}$ .*

**Corollary 4.9.** *For  $V \in \mathcal{O}_{\mathbb{Q}}$ , the claim of Corollary 4.4. remains valid for generic complex  $\hbar$ .*

*Proof.* This follows from Theorem 4.7 in the same way as Corollary 4.4 follows from Corollary 4.2.  $\square$

**Remark.** It is easy to generalize these results to the case when  $\mathfrak{g}$  is replaced with  $\tilde{\mathfrak{g}}'$ , using Theorem 4.5.

Now we would like to make some sharper statements, i.e. statements which hold for  $\hbar \notin \pi i\mathbb{Q}$ . To do this, we will assume for simplicity that  $\mathfrak{g}$  is a Kac-Moody algebra. In this case, it is known that the universal R-matrix is well defined outside of roots of unity, and that the nilpotent subalgebras of  $\mathcal{U}_{\hbar}$  have the same size as those for  $\mathfrak{g}$ . This allows to strengthen the above statements (using the same proofs) as follows.

**Theorem 4.10.** *If  $\hbar \notin \pi i\mathbb{Q}$ , then there exists a braided tensor functor  $F_{\hbar} : \mathcal{O}_{\mathbb{Q}} \rightarrow \mathcal{O}_{\mathbb{Q}, \hbar}$ , which is the identity functor at the level of  $\mathfrak{h}$ -graded vector spaces, and maps Verma modules to Verma modules and integrable modules to integrable modules.*

**Remark.** We expect that the functor  $F_{\hbar}$  is an equivalence if  $\hbar \notin \pi i\mathbb{Q}$ .

**Corollary 4.11.** *If  $V$  is in  $\mathcal{O}_{\mathbb{Q}}$ , then the claim of Corollary 4.4. remains valid for  $\hbar \notin \pi i\mathbb{Q}$ .*

**Remark.** Corollary 4.9 was proved by Drinfeld in the case when  $\mathfrak{g}$  is finite dimensional ([Dr3]). Corollary 4.11 for integrable modules was proved by Varchenko in [V].

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